Pascal’s Triangle

Pascal’s triangle is chock-full of delightful patterns, and is foundational for the study of binomial expansion, binomial probability theory and, more generally, the study of combinatorics (counting of combinations).

Pascal’s Triangle is a lattice of numbers where each number is the sum of the two directly above it. It starts with a single 1, and the first few rows look like this:

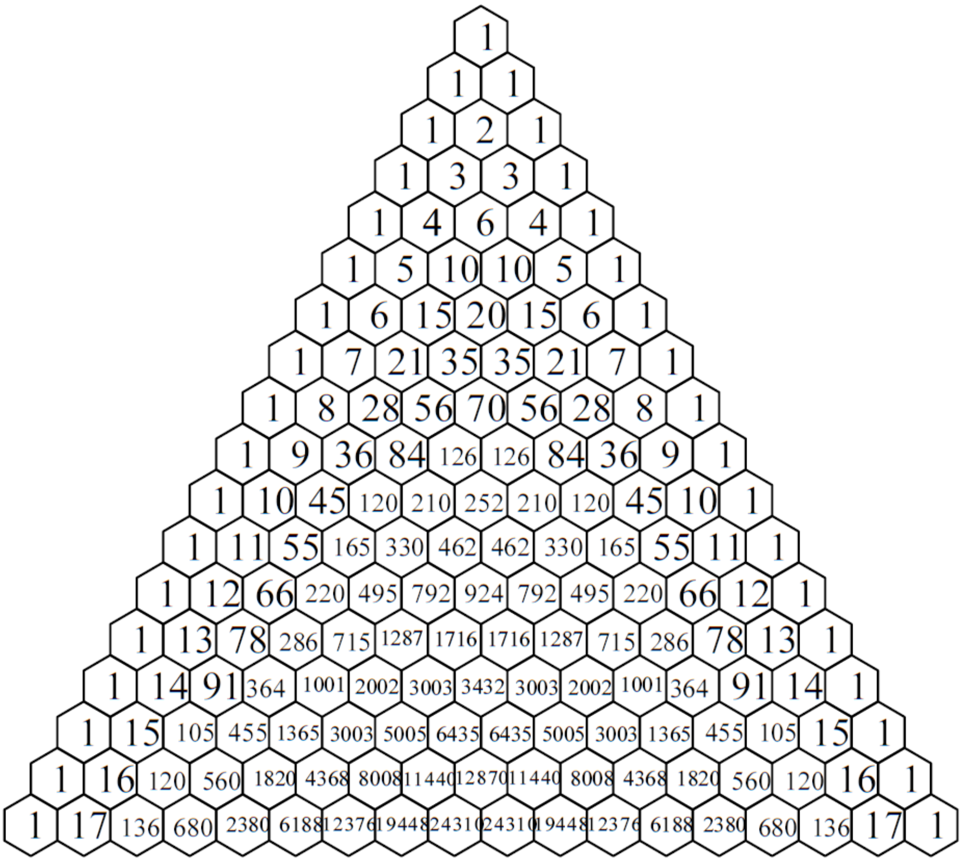
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
… … … … … …

Your turn:

You may have written out the first few terms of the triangle above many times. Have you tried it with modified starting values? Leaving off the first row, and using 1 2 for the next:
See what happens when you continue this pattern for a while:

1 2
1 3 2

One neat trick is to shade in all the even numbers in the original triangle. What do you get?



Turn over to see how to use Excel to generate Pascal’s Triangle …

Pascal's Triangle *with Excel*

Many mathematical processes are easy to describe and implement step-by-step, but can become a lot more challenging to describe in general. Excel will give us an easy way to repeatedly create lines of Pascal's triangle, and a bit of conditional formatting will allow us to investigate even more cool patterns when we're done.

	A	B	C
1	1		
2	=A1		
3			
4			
5			
6			

Start by entering a **1** in the top cell, **A1**. The blank cells are treated as containing a **0** by Excel, so that will work fine for us.

Next, create a formula below linking to the cell above: **=A1**. This column will form the left-hand edge of the triangle.

Note: our triangle will be right-angled rather than vertically symmetrical, but the patterns will still show.

	A	B	C
1	1		
2	1		
3	1		
4	1		
5	1		
6	1		
7	1		
8	=A7		

Copy the formula from cell **A2** down the column a way. Each cell should automatically refer to the one directly above it, not simply to **A1**. This is called *relative referencing*.

You can either click (once) on the cell, and use **CTRL+C** and **CTRL+V** to copy and paste, or hover the mouse over the small square in the bottom-right corner of the cell, click and drag.

	A	B	C
1	1		
2	1	=A1+B1	
3	1		
4	1		
5	1		
6	1		
7	1		
8	1		

Now for the proper Pascal's Triangle formula: in cell **B2** enter the formula **=A1+B1**. This will add the cells directly above and above-and-left, completing the second row.

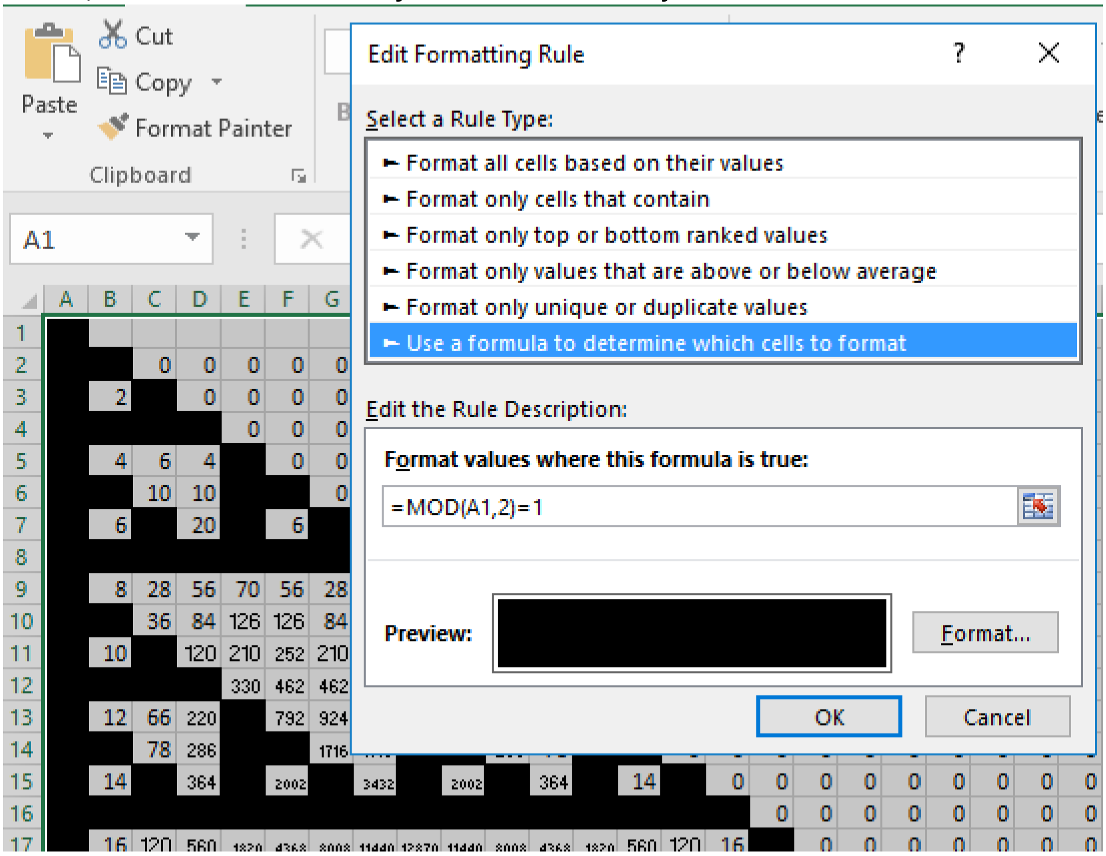
You can now copy this formula not just down, but across the sheet to the right as well.

You may also want to select the cells, right-click and choose Format Cells > Alignment and check Shrink to Fit so that all the numbers appear properly on the screen.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	1																
2	1	1															
3	1	2	1														
4	1	3	3	1													
5	1	4	6	4	1												
6	1	5	10	10	5	1											
7	1	6	15	20	15	6	1										
8	1	7	21	35	35	21	7	1									
9	1	8	28	56	70	56	28	8	1								
10	1	9	36	84	126	126	84	36	9	1							
11	1	10	45	120	210	252	210	120	45	10	1						
12	1	11	55	165	330	462	462	330	165	55	11	1					
13	1	12	66	220	495	792	924	792	495	220	66	12	1				
14	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1			
15	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1		
16	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1	
17	1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1
18	1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	136	17

If you want the zeroes to disappear, you can either delete them all by hand (somewhat tedious!) or just turn them all white using **conditional formatting**, but we're going to zoom out in a moment, so you won't see them anyway.

Conditional formatting allows us to colour cells based on their contents. Recall the modulo function? In Excel, using the function **=MOD(A1,2)** will return 0 if the contents of cell A1 is even, and 1 if not. Select all your cells, and choose **New Rule** under **Conditional Formatting**. Select the option **Use a formula to determine which cells to format**, and use the modulus function as shown below. *Note: if you get strange results, take care to make this formula a relative reference.*



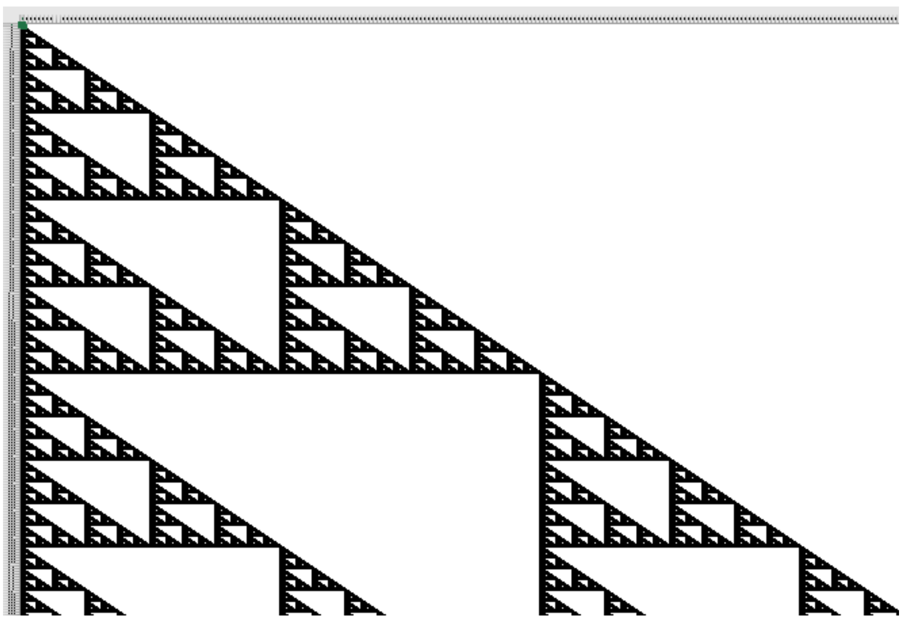
What you have created is a fractal! Copy and paste some more cells, and zoom out to get the full effect.

The reason things start to break down around row 45 is that Excel is struggling to deal with these large numbers. We can fix that by ignoring the values themselves, and simply entering a 0 or 1 in each cell.

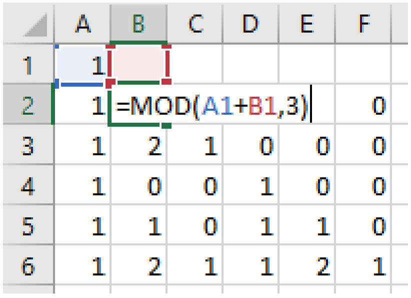
There are two ways to do this: modulo, or an **IF** function:

=MOD(A1+B1,2) or **=IF(A1=B1,0,1)**

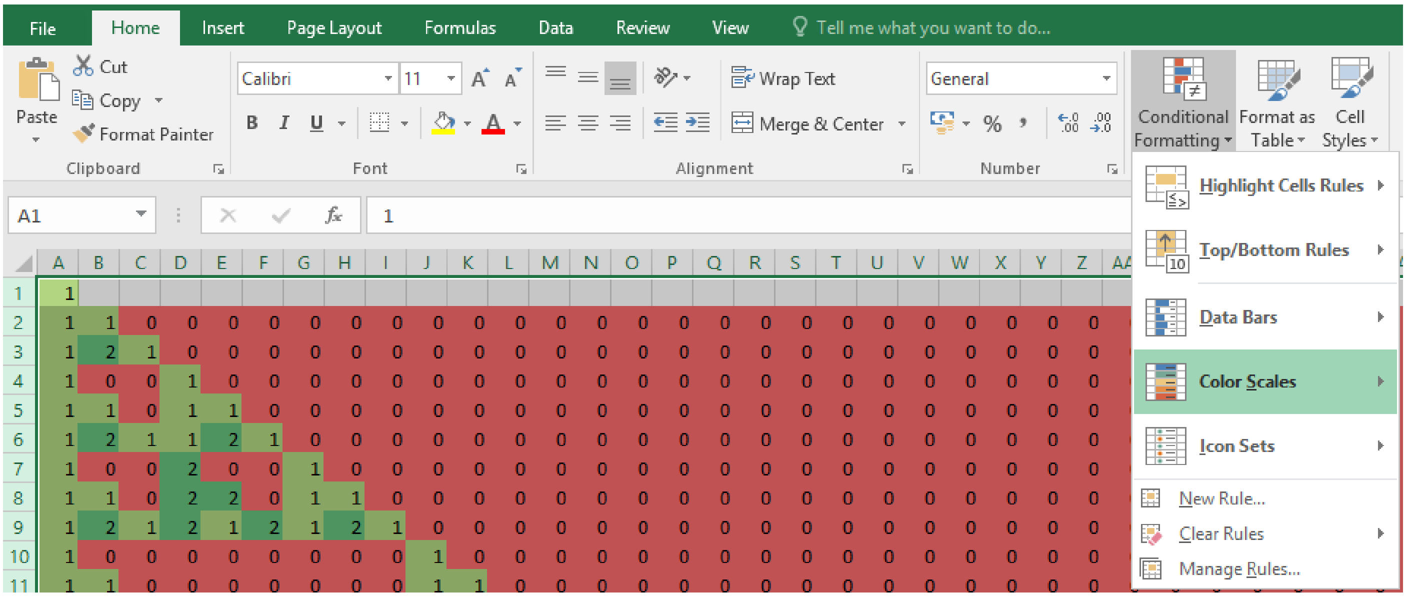
Can you see why these both do the same job?



What's next?



You can do similar things with modulo 3 rather than modulo 2. In fact, if you use the **MOD** function in your cells, the conditional formatting step becomes easier because you can use the built-in functions, and get some pretty colours:



Why does this happen?

Think about what happens when you add two even numbers, or two odd numbers, or one of each. Consider how each triangle within the pattern forms, and see what you notice about the rows where new triangles begin.

What else?

Look at the prime rows of Pascal's Triangle. Other than the 1 at the start and the end, every single value in the row divides by the prime. Why is that? Is there a way to prove it using the general formula for the k^{th} term in the n^{th} row of Pascal's triangle?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$